Operator Evolution via the Similarity Renormalization Group

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The Similarity Renormalization Group (SRG) provides a means to systematically evolve computationally difficult potentials and operators

\[
\frac{dO_s}{ds} = [\eta_s, O_s] = [[T_{rel}, H_s], O_s], \quad \iff \quad O_s = U_s O_{s=0} U_s^\dagger
\]

Where \( U_s = \sum_\alpha |\psi_\alpha(s)\rangle \langle \psi_\alpha(0)| \)

Hamiltonian Operator \( \rightarrow \) driven toward diagonal or decoupled form

\[
\lambda = \frac{1}{s^{1/4}} \text{ fm}^{-1}
\]

Can unitarily evolve operators consistent with any initial potential

Issues: – Decoupling – Many-body Operators – Basis issues – Factorization

\(^3S_1\ AV18\ Evolution\)
Demonstration of Decoupling In Expectation Values

- **Evolve** Hamiltonian & operators to \( \lambda \) in full space → **TRUNCATE** at \( \Lambda \):

- **Momentum distribution**
  - Calculated with **AV18 potential**.
  - \( \Lambda = 2.5 \text{ fm}^{-1} \)
  - \( \lambda = 6.0 \text{ fm}^{-1}, 3.0 \text{ fm}^{-1}, \text{ and } 2.0 \text{ fm}^{-1} \)

- **Decoupling** for all \( q \) is successful when \( \lambda < \Lambda \)

  ⇒ **Expectation values reproduced in truncated basis**
Many-body evolution with operators normal ordered in the vacuum:

\[ \frac{d\hat{O}_s}{ds} = [[T_{rel}, H_s], \hat{O}_s] \Rightarrow \left[ \sum_{ij} T_{ij} a_i^\dagger a_j, \sum_{i'j'} T_{i'j'} a_{i'}^\dagger a_{j'} + \frac{1}{2} \sum_{pqkl} V_{pqkl} a_p^\dagger a_q^\dagger a_l a_k + \cdots \right], \hat{O}_s \]

→ Only one non-vanishing contraction in the vacuum: \( a_i a_j^\dagger = \delta_{ij} \)

A general operator \( \hat{O} \) for an \( A \)-body system can be written as

\[ \hat{O} = \hat{O}^{(1)} + \hat{O}^{(2)} + \hat{O}^{(3)} + \cdots + \hat{O}^{(A)} \]

where the \( \hat{O}^{(i)} \) label the \( i = 1, 2, 3, \ldots, A \)-body components

– SRG operator \( \hat{O}_s \) will have contributions for all \( n \) so that \( \hat{O}^{(n)} \neq \hat{O}_s^{(n)} \).

Expanding commutators and making contractions, one finds:

→ Evolution of an operator is fixed in each \( n \)-particle subspace

→ How do we deal with this in practice?
\[ U_s = \sum_\alpha |\psi_\alpha(s)\rangle\langle\psi_\alpha(0)| \]

1-Particle Basis

2-Particle Basis

3-Particle Basis

\vdots

Evolution in n-Particle space determines n-Body component of operator for all \( A \)
SRG Evolution

\[ U_s = \sum_{\alpha} |\psi_\alpha(s)\rangle\langle\psi_\alpha(0)| \]

1-Particle Basis

2-Particle Basis

3-Particle Basis

\vdots

Evolution in n-Particle space determines n-Body component of operator for all A

A-Particle Calculations

Embed in A-Particle Basis

1-B Operator

2-B Operator

3-B Operator
SRG Evolution

\[ U_s = \sum_\alpha |\psi_\alpha(s)\rangle \langle \psi_\alpha(0)| \]

1-Particle Basis

2-Particle Basis

3-Particle Basis

\[ U_s^{(1)} \]

\[ U_s^{(2)} \]

\[ U_s^{(3)} \]

\[ \vdots \]

Evolution in n-Particle space determines n-Body component of operator for all A

Embed in A-Particle Basis

**Original Operator**

\[ \hat{O} = \hat{O}^{(1)} + \hat{O}^{(2)} + \cdots \]

\[ U_s^{(1)} \hat{O} U_s^{(1)} \]

1-B Operator

2-B Operator

3-B Operator

A-Particle Calculations
SRG Evolution

\[ U_s = \sum_\alpha \langle \psi_\alpha (s) | \psi_\alpha (0) \rangle \]

1-Particle Basis

2-Particle Basis

3-Particle Basis

\vdots

Evolution in n-Particle space determines n-Body component of operator for all A

A-Particle Calculations

Original Operator

\[ \hat{O} = \hat{O}^{(1)} + \hat{O}^{(2)} + \cdots \]

Embed in 2-Particle Basis

\[ U_s^{(1)} \hat{O} U_s^{(1)} \]

1-B Operator

Embed in A-Particle Basis

\[ U_s^{(2)} \hat{O} U_s^{(2)} \]

2-B Operator

\[ \hat{O}_s^{(1)} \]

3-B Operator
SRG Evolution

\[ U_s = \sum_\alpha |\psi_\alpha(s)\rangle \langle \psi_\alpha(0) | \]

1-Particle Basis

2-Particle Basis

3-Particle Basis

\ldots

Evolution in n-Particle space determines n-Body component of operator for all A

A-Particle Calculations

Original Operator

\[ \hat{O} = \hat{O}^{(1)} \]

\[ + \hat{O}^{(2)} \]

\[ + \ldots \]

Embed in 2-Particle Basis

1-B Operator

Embed in 3-Particle Basis

2-B Operator

Embed in A-Particle Basis

3-B Operator
Embedding Other Operators

Consider the following momentum distributions in 1D model . . .

Problem

Corrected

Solution: Operators must be “boosted” to embed into A-particle space

- Revised Embedding Process . . .
Operator Evolution & Extraction Process

SRG Evolution

\[ U_s = \sum_\alpha |\psi_\alpha(s)\rangle\langle\psi_\alpha(0)| \]

1-Particle Basis

\[ U_s^{(1)} \]

2-Particle Basis

\[ U_s^{(2)} \]

3-Particle Basis

\[ U_s^{(3)} \]

Evolution in n-Particle space determines n-Body component of operator for all A

A-Particle Calculations

Original Operator

\[ \hat{O} = \hat{O}^{(1)} + \hat{O}^{(2)} + \cdots \]

Embed in 2-Particle Basis

\[ U_s^{(1)} \hat{O} U_s^{(1)} \]

1-B Operator

Embed in 3-Particle Basis

\[ U_s^{(2)} \hat{O} U_s^{(2)} \]

2-B Operator

\[ \hat{O}_s^{(1)} + \hat{O}_s^{(2)} + \cdots \]

Embed in A-Particle Basis

3-B Operator
SRG Evolution

\[ U_s = \sum_\alpha |\psi_\alpha(s)\rangle \langle \psi_\alpha(0)| \]

1-Particle Basis

2-Particle Basis

3-Particle Basis

... Evolution in n-Particle space determines n-Body component of operator for all A

A-Particle Calculations

Original Operator

\[ \hat{O} = \hat{O}^{(1)} + \hat{O}^{(2)} + \cdots \]

Embed in 2-Particle Basis

\[ \hat{O}_s^{(1)} \]

Embed in 3-Particle Basis

\[ \hat{O}_s^{(1)} + \hat{O}_s^{(2)} + \cdots \]

Embed in A-Particle Basis

1-B Operator

2-B Operator

3-B Operator

Boost

Boost

Boost
High and Low Momentum operators in the Deuteron

- **Integrand** of $\langle \psi_d | U^\dagger (U a_q^\dagger a_q U)^\dagger U | \psi_d \rangle$ for $q = 0.34 \text{ fm}^{-1}$

- **Momentum Distribution**

- **Integrand** for $q = 3.02 \text{ fm}^{-1}$

- **Decoupling** ↔ High momentum components suppressed
  - Integrated value does not change, but nature of operator does
  - Similar for other operators: $\langle r^2 \rangle, \langle Q_d \rangle, \langle \frac{1}{r} \rangle, \langle G_C \rangle, \langle G_Q \rangle, \langle G_M \rangle$, etc.
High and Low Momentum Operators in 1D Oscillator Basis

- **A=3, 1D model boson system**
- **Evolve Hamiltonian & operators to \( \lambda \) at large \( N_{\text{max}} \)**
  - Truncate model space at \( N_{\text{cut}} \)
  - Poor convergence of long range operators

\[
\Lambda_{UV} \sim \sqrt{m N_{\text{max}} \hbar \omega}; \quad \Lambda_{IR} \sim \sqrt{\frac{m \hbar \omega}{N_{\text{max}}}}
\]

- Can this be corrected?

- Number operator at: \( q = 15.0 \) & \( q = 0.5 \)
An alternative generator - diagonal in the basis

- The harmonic oscillator Hamiltonian $H_{\text{ho}}$ is diagonal in the basis.
- Consider a naive choice (replace $T_{\text{rel}}$ with $H_{\text{ho}}$ in SRG generator):

$$\eta_s = [H_{\text{ho}}, H_s] \longrightarrow \frac{dO_s}{ds} = [\eta_s, O_s]$$

- Greatly improved convergence
- Generates spurious deep bound states . . .
Controlled IR and UV renormalization

Consider $T_{\text{rel}} + \alpha r^2$, where $\alpha$ is a parameter which can be adjusted to optimize the renormalization (here, $\alpha = 1$), so that

$$\eta_s = [T_{\text{rel}} + \alpha r^2, H_s]$$

- Convergence improves with decreasing $\lambda$
- No spurious deep bound states. Is hierarchy of many-body forces under control?
**Factorization in Few-Body Nuclei**

- **Variational Monte Carlo Calculation**
  → Using AV14 NN potential

- **1D few-body HO space calculation**
  → System of $A$ bosons interacting via a model potential


- Possible explanation of scaling behavior
  → Results from dominance of NN potential and short-range correlations (Frankfurt, et al.)

→ A Test Bed for 3D NCSM calculations:

- Alternative explanation of scaling behavior
  → Results from *factorization* . . .
\[ \langle \psi_d | U_\lambda a_q^+ a_q U_\lambda^\dagger | \psi_d \rangle \text{ is independent of } \lambda. \text{ What is the nature of } U_\lambda a_q^+ a_q U_\lambda^\dagger? \]

- **From Decoupling:** write
  \[ \langle \psi_\lambda | U_\lambda \hat{O} U_\lambda^\dagger | \psi_\lambda \rangle \cong \int_0^\lambda dk' \int_0^\infty dq' \int_0^\infty dq \int_0^\lambda dk \, \psi_\lambda^\dagger(k') U_\lambda(k', q') \hat{O}(q', q) U_\lambda(q, k) \psi_\lambda(k) \]

- **Using Factorization:** set \( U_\lambda(k, q) \rightarrow K_\lambda(k) Q_\lambda(q) \), where \( k < \lambda \) and \( q \gg \lambda \).
  \[ \Rightarrow \int_0^\lambda \int_0^\lambda \psi_\lambda^\dagger(k') \left[ \int_0^\lambda \int_0^\lambda U_\lambda(k', q') \hat{O}(q', q) U_\lambda(q, k) + I_{QOQ} K_\lambda(k') K_\lambda(k) \right] \psi_\lambda(k) \]
  
  - **Low Momentum Structure**
  
  where \( I_{QOQ} = \int_\lambda^\infty dq' \int_\lambda^\infty dq \left[ Q_\lambda(q') \hat{O}(q', q) Q_\lambda(q) \right] \)
  
  - **Universal**

- **Valid** when initial operators weakly couple high and low momentum, e.g.,

\[ r^2 \quad G_C(q = 3.02 \text{ fm}^{-1}) \quad a_q^+ a_q(q = 3.02 \text{ fm}^{-1}) \]
**Factorization in Few-Body Nuclei**

- **Variational Monte Carlo Calculation**
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- **Possible explanation of scaling behavior**
  - Results from dominance of NN potential and short-range correlations (Frankfurt, et al.)

- **1D few-body HO space calculation**
  - System of $A$ bosons interacting via a model potential

- **Alternative explanation of scaling behavior**
  - Results from factorization

\[ \int_0^\lambda \int_0^\lambda \psi_{\lambda}^\dagger(k') [I_{QOQ} K_{\lambda}(k') K_{\lambda}(k)] \psi_{\lambda}(k) \]
Recent Progress:
- Consistently evolved nuclear operators with SRG in deuteron & model 1D few-body calculations
- Extraction & embedding process for few-body operators formulated and tested
- Explored alternative generators for oscillator basis
- Factorization demonstrated for few-body model calculation

Computational Issues:
- SRG evolution in n-particle basis
  - Exponential growth of matrix size

Plan:
- Establish bounds on growth of many-body operators
- Do calculations in 3D in harmonic oscillator basis
- Explore factorization of other operators (e.g., electroweak)
The End